LITERATURE CITED

1. Kh. N. Sui, Izv. Akad. Nauk Est. SSR, Tekh. Fiz. Mat. Nauk, 11, No. 4, 215-222 (1962).
2. A. N. Sekundov, Investigation of Turbulent Jets of Air, Plasma, and Real Gas [in Russian], (1967), pp. 131-142.
3. Kh. N. Sui and Yu. V. Ivanov, Izv. Akad. Nauk Est. SSR Tekh. Fiz. Mat. Nauk, 8, No. 2, 88-96 (1959).
4. M. A. Laryushkin and V. P. Knyaz'kov, Izv. Vyssh. Uchebn. Zaved. Énerg., No. 12, 83-86 (1983).
5. L. A. Vulis and T. P. Leont'ev, Izv. Akad. Nauk Kaz. SSSR, Ser. Énerg., 9, 109-122 (1955).
6. A. S. Ginevskii, Prom. Aérodin., 23, 96-104 (1962).

GRADIENT NUMERICAL-ANALYTICAL METHOD FOR SOLUTION OF THE NAVIER–STOKES EQUATIONS FOR A VISCOUS INCOMPRESSIBLE FLUID

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A method is presented for solution of the Navier–Stokes equations in an extremal formulation based on a joint application of Pontryagin's maximum principle and a representation of the unknown functions in the form of power series.

The system of Navier–Stokes equations describing plane laminar motion of a viscous incompressible fluid in velocity–pressure variables has the form

$$\frac{du}{\partial t}+u\frac{\partial u}{dx}+v\frac{\partial u}{\partial y}=-\frac{\partial P}{\partial x}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \tag{1}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial P}{\partial y}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right), \tag{2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{3}$$

In solving the system (1)-(3), as a rule, one replaces the continuity equation by the Poisson equation for the pressure. It was noted in [1] that the main difficulty in this case consists in integrating the Poisson equation with Neumann type boundary conditions (as the boundary condition one uses an expression for projection of momentum on a wall). A part of the algorithms for solving the Navier–Stokes equations is connected with their integration in stream function-vorticity variables. Moreover, specific difficulties arise in calculation of the boundary conditions for the vorticity, which are not given from the physical conditions. We present below an iterational algorithm for solving Eqs. (1)-(3) which makes use of the natural boundary conditions and does not require integration of the Poisson equation.

We introduce the functional

$$J_0=\int\limits_0^{t_\mathrm{K}}\left[\oint\limits_\Gamma u_n ds\right]^2 dt. \tag{4}$$

By decomposing the region with the moving fluid into elementary regions bounded by lines joining nodes of a grid, we can replace the problem of finding the pressure from the condition for a minimum of functional (4) for each elementary region by the solution of Eqs. (1), (2). As a result, along with the pressure we shall determine the velocity components.

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As an example illustrating this algorithm, we consider the problem of determining the fluid flow parameters in a square cavity with an upper cover moving with constant velocity. Initial and boundary conditions in this case have the form:

$$u(x,\ y,\ 0)=0, \qquad v(x,\ y,\ 0)=0, \tag{5}$$

$$u(0,\ y,\ t)=u(1,\ y,\ t)=u(x,\ 0,\ t)=0, \quad u(x,\ 1,\ t)=1, \tag{6}$$

$$v(0,\ y,\ t)=v(1,\ y,\ t)=v(x,\ 0,\ t)=v(x,\ 1,\ t)=0. \tag{7}$$

To integrate Eqs. (1), (2) with conditions (5)-(7) with a given pressure field, we employ the numerical-analytical method from [2], allowing us to reduce the solution of the initial-boundary problem to the integration of a system of ordinary differential equations (SODE) in Cauchy form. We decompose the region of fluid flow by means of a uniform grid $x_m = 1/M(m-1)$, $m = \overline{2, M}$, $y_\ell = 1/L(\ell - 1)$, $\ell = \overline{2, L}$. In a neighborhood of each grid node (m, ℓ) we shall seek the velocity components and the pressure in the series form

$$\begin{Bmatrix} u(x,\ y,\ t) \\ v(x,\ y,\ t) \\ P(x,\ y,\ t) \end{Bmatrix} = \sum_{n=1}^{\infty} \begin{Bmatrix} u_n(y,\ t) \\ v_n(y,\ t) \\ P_n(y,\ t) \end{Bmatrix} \left(\frac{x-x_m}{x_{m+1}-x_m}\right)^{n-1},$$

$$\begin{Bmatrix} u_n(y,\ t) \\ v_n(y,\ t) \\ P_n(y,\ t) \end{Bmatrix} = \sum_{k=1}^{\infty} \begin{Bmatrix} u_n^k(t) \\ v_n^k(t) \\ P_n^k(t) \end{Bmatrix} \left(\frac{y-y_l}{y_{l+1}-y_l}\right)^{k-1}. \tag{8}$$

Substituting Eqs. (8) into Eqs. (1), (2) and equating coefficients of identical powers of the spatial variables, we obtain

$$\frac{du_n^k}{dt} = f_n^k = -F_n^k + \frac{1}{\mathrm{Re}}\left\{\frac{(n+1)\,n}{(x_{m+1}-x_m)^2}\,u_{n+2}^k + \frac{(k+1)\,k}{(y_{l+1}-y_l)^2}\,u_n^{k+2}\right\}, \tag{9}$$

$$\frac{dv_n^k}{dt} = g_n^k = -G_n^k + \frac{1}{\mathrm{Re}}\left\{\frac{(n+1)\,n}{(x_{m+1}-x_m)^2}\,v_{n+2}^k + \frac{(k+1)\,k}{(y_{l+1}-y_l)^2}\,v_n^{k+2}\right\}, \tag{10}$$

where

$$F_n^k = \frac{n}{x_{m+1}-x_m}\,P_{n+1}^k + \sum_{i=1}^{n}\sum_{j=1}^{k}\left\{\frac{i}{(x_{m+1}-x_m)}\,u_{i+1}^j\,u_{n-i+1}^{k-j+1} + \frac{j}{y_{l+1}-y_l}\,u_i^{j+1}\,v_{n-i+1}^{k-j+1}\right\},$$

$$G_n^k = \frac{k}{y_{l+1}-y_l}\,P_n^{k+1} + \sum_{i=1}^{n}\sum_{j=1}^{k}\left\{\frac{i}{x_{m+1}-x_m}\,v_{i+1}^j\,u_{n-i+1}^{k-j+1} + \frac{j}{y_{l+1}-y_l}\,v_i^{j+1}\,v_{n-i+1}^{k-j+1}\right\}.$$

We consider a computational scheme obtained from relations (9) and (10) with n = 1, 2, k = 1, 2. To determine the values of u_3^k, u_4^k, k = 1, 2 we use the equations

$$\sum_{n=1}^{4} u_{n,m}^{k,l} = u_{1,m+1}^{k,l}, \tag{11}$$

$$\sum_{n=1}^{4} (-1)^{n-1} u_{n,m}^{k,l} = u_{1,m-1}^{k,l}, \tag{12}$$

whence

$$u_{3,m}^{k,l} = \frac{1}{2}\left[u_{1,m+1}^{k,l} + u_{1,m-1}^{k,l}\right] - u_{1,m}^{k,l}, \tag{13}$$

$$u_{4,m}^{k,l} = \frac{1}{2}\left[u_{1,m+1}^{k,l} - u_{1,m-1}^{k,l}\right] - u_{2,m}^{k,l}. \tag{14}$$

In a similar way we find $u_{n,m}^{3,\ell}$, $u_{n,m}^{4,\ell}$, $v_{n,m}^{3,\ell}$, $v_{n,m}^{4,\ell}$, n = 1, 2, and $v_{3,m}^{k,\ell}$, $v_{4,m}^{k,\ell}$, k = 1, 2. Since initially the fluid is quiescent, the initial conditions for the SODE (9) and (10) have the form

$$u_{n,m}^{k,l}(0) = 0,\ v_{n,m}^{k,l}(0) = 0,\ n = \overline{1,\ \infty},\ k = \overline{1,\ \infty}. \tag{15}$$

Thus, for a given pressure field, we can determine the velocity components by integrating the SODE (9), (10), taking into account expressions of the form (13), (14) and the initial conditions (15). We substitute Eqs. (8) into relation (4) and take the integral over the boundary of the region $[x_{m-1},\ x_{m+1}] \times [y_{\ell-1},\ y_{\ell+1}]$ containing the node (m, ℓ):

$$J_{0,m}^{l} = \int_0^{t_{\text{K}}} [j_m^l]^2\,dt, \tag{16}$$

where

$$j_m^l = u_{1,m+1}^{1,l} + \frac{1}{3}\,u_{1,m+1}^{3,l} - u_{1,m-1}^{1,l} - \frac{1}{3}\,u_{1,m-1}^{3,l} + \frac{x_{m+1} - x_m}{y_{l+1} - y_l} \times$$
$$\times \left[v_{1,m}^{1,l+1} + \frac{1}{3}\,v_{3,m}^{1,l+1} - v_{1,m}^{1,l-1} - \frac{1}{3}\,v_{3,m}^{1,l-1} \right].$$

Finally, we can formulate the stated problem in the following way: It is required to so select the values of the components of the pressure function $P_{n,m}^{k,\ell}(t)$, so that the components of the velocity vector will minimize the value of the functional

$$J_0 = \int_0^{t_{\text{K}}} \sum_{m=2}^{M} \sum_{l=2}^{L} [j_m^l]^2\,dt. \tag{17}$$

To do this we use the Pontryagin maximum principle [3] and one of the gradient methods of optimization. We form the function

$$H(u_{n,m}^{k,l},\ v_{n,m}^{k,l},\ U_{n,m}^{k,l},\ V_{n,m}^{k,l},\ P_{n,m}^{k,l},\ t) = \sum_{m=2}^{M} \sum_{l=2}^{L} \left\{ -[j_m^l]^2 + \sum_{n=1}^{2} \sum_{k=1}^{2} [U_{m,n}^{k,l} f_{n,m}^{k,l} + V_{n,m}^{k,l} g_{n,m}^{k,l}] \right\}. \tag{18}$$

Consequently, with the fundamental controlling SODE (9) and (10) we can associate the adjoint system

$$\frac{dU_{n,m}^{k,l}}{dt} = -\frac{\partial H}{\partial u_{n,m}^{k,l}},\quad n = 1,\ 2,\ k = 1,\ 2,\ m = \overline{2,\ M},\ l = \overline{2,\ L}, \tag{19}$$

$$\frac{dV_{n,m}^{k,l}}{dt} = -\frac{\partial H}{\partial v_{n,m}^{k,l}},\ n = 1,\ 2,\ k = 1,\ 2,\ m = \overline{2,\ M},\ l = \overline{2,\ L}. \tag{20}$$

Since no restrictions are imposed on the values of the velocity components at a finite time instant $t = t_K$, the initial condition for the SODE (19), (20) is

$$U_{n,m}^{k,l}(t_{\text{K}}) = 0,\quad V_{n,m}^{k,l}(t_{\text{K}}) = 0. \tag{21}$$

To determine the maximum of function (18) with respect to $P_{n,m}^{k,\ell}(t)$ by a gradient method, we need expressions for calculation of the corresponding derivatives. In this case, they assume the following form:

$$\frac{\partial H}{\partial P_{n,m}^{k,l}} = \frac{n-1}{x_{m+1} - x_m}\,U_{n-1,m}^{k,l} + \frac{k-1}{y_{l+1} - y_l}\,V_{n,m}^{k-1,l},\ n = 1,\ 2,\ k = 1,\ 2, \tag{22}$$

$$\frac{\partial H}{\partial P_{3,m}^{k,l}} = \frac{2}{x_{m+1} - x_m}\,U_{2,m}^{k,l},\ k = 1,\ 2, \tag{23}$$

$$\frac{\partial H}{\partial P_{n,m}^{3,l}} = \frac{2}{y_{l+1} - y_l}\,V_{n,m}^{2,l},\ n = 1,\ 2. \tag{24}$$

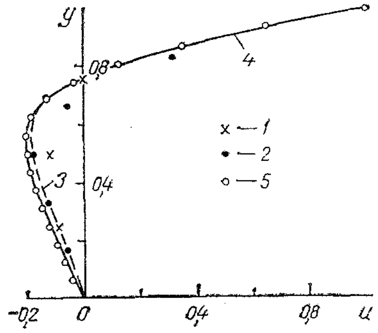

Fig. 1. Distribution of horizontal velocity component at section x = 0.5. Labels 1, 2, 3, and 4 correspond to M = L values of 4, 6, 8, and 10, respectively; points labelled 5 are the result of solving the problem in stream function-vorticity variables.

In organizing the iterational process for the search for a maximum, we employ the conjugate gradient method, a method well-suited for the solution of inverse heat conduction problems in an extremal formulation [4] where the values of the pressure components at the p-th iteration are determined from the expressions

$$\{P_{n,m}^{k,l}\}^{p+1} = \{P_{n,m}^{k,l}\}^{p} - \alpha_p \{\Phi_{n,m}^{k,l}\}^{p},\ p = 0,\ 1,\ \ldots \tag{25}$$

where

$$\{\Phi_{n,m}^{k,l}\}^{p} = -\left\{\frac{\partial H}{\partial P_{n,m}^{k,l}}\right\}^{p} + \beta_p \{\Phi_{n,m}^{k,l}\}^{p-1},\quad \beta_0 = 0,$$

$$\beta_p = \frac{\left[\left\{\frac{\partial H}{\partial P_{n,m}^{k,l}}\right\}^{p} - \left\{\frac{\partial H}{\partial P_{n,m}^{k,l}}\right\}^{p-1}\right]\left\{\frac{\partial H}{\partial P_{n,m}^{k,l}}\right\}^{p}}{\left[\left\{\frac{\partial H}{\partial P_{n,m}^{k,l}}\right\}^{p-1}\right]^2}.$$

The step multiplier at each iteration is obtained from the condition

$$\alpha_p = \min_{\alpha \geqslant 0} J_0 [\{P_{n,m}^{k,l}\}^{p} - \alpha \{\Phi_{n,m}^{k,l}\}^{p}]. \tag{26}$$

Based on the algorithm developed for solving the Navier–Stokes equations, we carried out calculations for determining the velocity and pressure for the flow of a fluid in a square cavity with a uniformly moving upper cover, Re = 1. In Fig. 1 we show, for the solution obtained, the resulting stationary distributions of the horizontal velocity component at the section x = 0.5 corresponding to calculations on uniform grids with various numbers of internal nodes. As the grid becomes more dense the difference between corresponding results decreases, and a comparison of the results computed on the least dense gridwork with the solution of the same problem in the stream function-vorticity variables shows good agreement. On the whole, our calculations confirm the effectiveness of our approach to solving the Navier–Stokes equations, wherein use can be made of natural boundary conditions for the fluid velocity and there is no need in determining the pressure to solve a Neumann problem for the Poisson equation.

NOTATION

x, y, space coordinates; t, time; u, v, horizontal and vertical components of velocity; P, pressure; Re, Reynolds number; Γ, boundary of region; u_n, projection of velocity vector on normal to boundary of region; s, arc length of integration contour; t_K, finite time instant; m, ℓ, number of grid nodes; M, L, parameters determining the number of nodes of a gridwork; n, k, power series indices; J_0, minimizing functional; U, V, conjugate functions; α, step multiplier of the conjugate gradient method; p, iteration number.

LITERATURE CITED

1. P. J. Roache, Computational Hydronamics, Prentice-Hall (1976).
2. A. A. Shmukin, Applied Problems in the Aerodynamics of Aircraft [in Russian], Kiev (1984), pp. 87-93.
3. L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mischenko, Mathematical Theory of Optical Processes [in Russian], Moscow (1983).
4. O. M. Alifanov, Identification of Aircraft Heat Exchange Processes [in Russian], Moscow (1979).